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A general scheme for obtaining graviton spectrums

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Abstract

The aim of this contribution is to present a general scheme for obtaining graviton spectra from modified gravity theories, based on a theory developed by Grishchuk in the mid 1970s. We try to be pedagogical, putting in order some basic ideas in a compact procedure and also giving a review of the current trends in this arena. With the aim to fill a gap for the interface between quantum field theorists and observational cosmologist in this matter, we highlight two interesting applications to cosmology: clues as to the nature of dark energy; and the possibility of reconstruction of the scalar potential in scalar–tensor gravity theories.

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1. Introduction

There should be a cosmological background of gravitons whose existence is predicted by many metric theories of gravity, including Einstein's general relativity theory. We will sketch the general method for computing the graviton production arising from a general theory of gravity—not specifically Einstein's theory—keeping the formalism as much general as possible in order to be applicable to whatever theory of gravity of interest. We will end by highlighting two interesting applications to cosmology.

2. Graviton's spectrum computation

We will focus our attention on the graviton production in different transition epochs during the universe evolution, with gravity described by a general Lagrangian density, and the metric of spacetime described by the usual Friedman–Lemaître–Robertson–Walker (FLRW) line element. In this context, the generation of gravitons arises from the amplification of vacuum fluctuations during the transition epochs. This mechanism was first discussed by Grishchuk in the middle 1970s [1], and Bogoliubov established a formalism in this subject named after him (see [2] for details).

We will use, throughout this paper unless otherwise stated, physical units $8\pi G_N = c = \hbar = 1$.

We start from a general Lagrangian in four dimensions for maximally symmetric spacetimes, where it is always possible to put the Riemann Tensor $\Re_{\alpha\beta\mu\nu}$ in terms of the scalar curvature $\Re$ alone so that

$$S = \int \mathrm{d}^4 x \sqrt{-g}\{f(\Re, \partial_\mu \Re, \Box \Re, g_{\mu\nu}) + \mathcal{L}_{\text{Matter}}\}. \tag{1}$$

From this, the general procedure for computing background metric perturbations (i.e., gravitons) can be either to perturb directly the action to second order in the background metric perturbation, or to derive first the equations of motion and perturb them after to first order in the metric perturbation. We will proceed by the second method and we will assume also a maximally symmetric spacetime so that our Lagrangian is a function of the Ricci scalar alone. In this case, the *Field equations* are

$$f'(\Re)\Re_{\alpha\beta} - \tfrac{1}{2} f(\Re) g_{\alpha\beta} = f'(\Re)^{;\mu\nu}(g_{\alpha\mu}g_{\beta\nu} - g_{\alpha\beta}g_{\mu\nu}) + \widetilde{T}^{\text{Matter}}_{\alpha\beta} \tag{2}$$

where the prime "$'$", unless otherwise stated, indicates the derivative with respect to the Ricci scalar $\Re$.

In the case $f(\Re) = \Re + 2\Lambda$, we recover general relativity with the *cosmological constant* Λ. It is also worthwhile to point out that in order to have constant curvature solutions $R = R_0$, the *Lagrangian* $f(\Re)$ must satisfy the condition $f'(\Re_0) = 2f(\Re_0)/\Re_0$ (the *on-shell* condition).

In the case of FLRW metrics the general action equation (1) can be recast into an action whose field equations are a generalization of the usual Friedmann–Lemaître field equations in general relativity, using the scale factor a and the Ricci scalar $\Re$ as the canonical variables, in a way that is common use in canonical quantization techniques,

$$2\left(\frac{\ddot{a}}{a}\right) + \left(\frac{\dot{a}}{a}\right)^2 + \frac{k}{a^2} = -P_{\text{Total}} \tag{3}$$

$$f''(\Re)\left\{\Re + 6\left[\left(\frac{\ddot{a}}{a}\right) + \left(\frac{\dot{a}}{a}\right)^2 + \frac{k}{a^2}\right]\right\} = 0 \tag{4}$$

constrained by the following energy condition $\left(\frac{\dot{a}}{a}\right)^2 + \frac{k}{a^2} = \frac{1}{3}\rho_{\text{Total}}$. Using this last, we can rewrite equation (3) as $\left(\frac{\dot{a}}{a}\right)^2 + \frac{k}{a^2} = \frac{1}{3}\rho_{\text{Total}}$. From this, we obtain the conditions for the accelerated and decelerated behaviours: $\rho_{\text{Total}} + 3P_{\text{Total}} < 0 \Rightarrow$ Acceleration and $\rho_{\text{Total}} + 3P_{\text{Total}} > 0 \Rightarrow$ Deceleration, where we define the *total pressure and density* as $P_{\text{Total}} = P_{\text{Matter}} + P_{\text{Curvature}}$ and $\rho_{\text{Total}} = \rho_{\text{Matter}} + \rho_{\text{Curvature}}$, where

$$P_{\text{Curvature}} \equiv \frac{1}{f'(\Re)}\left\{2\left(\frac{\dot{a}}{a}\right)\dot{\Re} f''(\Re) + \ddot{\Re} f''(\Re) + \dot{\Re}^2 f'''(\Re) - \frac{1}{2}[f(\Re) - \Re f'(\Re)]\right\} \tag{5}$$

$$\rho_{\text{Curvature}} \equiv \frac{1}{f'(\Re)}\left\{\frac{1}{2}[f(\Re) - \Re f'(\Re)] - 3\left(\frac{\dot{a}}{a}\right)\dot{\Re} f''(\Re)\right\}. \tag{6}$$

From that one defines the so-called *Barotropic index* w as the ratio between the pressure and the density for each the matter and the curvature terms.

As many observations indicate—high redshift supernovae surveys; CMB anisotropies; Sunyaev–Zeldovich/ x-ray methods, and others—at the present epoch, the Universe is experiencing a period of accelerated expansion. Assuming that all matter components contribute with a non-negative pressure, then we have that $\rho_{\text{Curvature}} > \frac{1}{3}\rho_{\text{Total}}$.

We will consider a spatially flat FLRW metric (the relation between *cosmic time* t and *conformal time* η is given by $\mathrm{d}t = a\,\mathrm{d}\eta$):

$$\mathrm{d}s^2 = \mathrm{d}t^2 - a^2(t)\,\mathrm{d}\vec{x}^{\,2} = a^2(\eta)[\mathrm{d}\eta^2 - \mathrm{d}\vec{x}^{\,2}]. \tag{7}$$

To first order, a transverse, traceless metric perturbation of a background metric that represents sourceless and weak gravitational waves can be described as

$$h_{ij}(\overline{x}) = \sum\nolimits_{l:1,2}[A(\vec{k},l)h_{ij}(\overline{x},\vec{k},l) + A^+(\vec{k},l)h^*_{ij}(\overline{x},\vec{k},l)]\,\mathrm{d}^3k \tag{8}$$

where $\overline{x} \equiv (\eta, \vec{x})$; i, j: 1, 2, 3 are the spatial indices; l: 1, 2 indicate the two polarization states for the gravitational wave; + denotes the transposed and $*$ indicates complex conjugation. In the above expression the following functions are defined:

$$h_{ij}(\overline{x},\vec{k},l) \equiv h(\overline{x},\vec{k})\varepsilon_{ij}(\vec{k},l) \qquad h(\overline{x},\vec{k}) \equiv \frac{\sqrt{16\pi g}}{(2\pi)^{\frac{3}{2}}}\frac{\mu(k,\eta)}{R(\eta)}\,\mathrm{e}^{\mathrm{i}\vec{k}\,\vec{x}};$$
$$R \equiv a\left[\frac{\partial f(\Re)}{\partial \Re}\right]^{\frac{1}{2}}. \tag{9}$$

The constant g appearing in the above is a *generalized gravitational constant* reducing to the usual G_N in the case of general relativity. We point out that, also in general relativity, the function R reduces to $R = a$, as $f(\Re) \propto \Re$. The functions $\varepsilon_{ij}(\vec{k},l)$ are polarization tensors satisfying $\varepsilon_{ij}(\vec{k},l)\varepsilon^{ij}(\vec{k},l') = 2\delta_{ll'}$ and $\varepsilon_{ij}(\overrightarrow{-k},l) = \varepsilon_{ij}(\vec{k},l)$. The functions $\mu(k,\eta)$ are called *mode functions*. Inserting the metric perturbation equation (8) into the generalized equations of motion, one arrives at an equation for these mode functions,

$$\mu''(k,\eta) + \left[k^2 - \frac{R''(\eta)}{R(\eta)}\right]\mu(k,\eta) = 0, \tag{10}$$

which reminds us of the Schrödinger equation; here the prime "′" indicates derivatives with respect the conformal time η. It is usual to define the following quantity $\frac{\mu(k,\eta)}{R(\eta)} \equiv Y(k,\eta)$.

In the classical theory, the functions $A(\vec{k},l)$ and $A^+(\vec{k},l)$ would be complex constants. In the quantized theory they are *annihilation* and *creation* operators respectively, satisfying the usual *canonical commutation relations* $[A(\vec{k},l), A^+(\vec{k'},l')] = \delta(\vec{k},\vec{k'})\delta_{ll'}$ and $[A(\vec{k},l), A(\vec{k'},l')] = 0 = [A^+(\vec{k},l), A^+(\vec{k'},l')]$.

We use the following conditions on the normalization factor and on the Wronskian of the solutions $\mu(k,\eta)$ for equation (10), *Normalization factor* $= (2\pi)^{-3/2}$ and $\mu(k,\eta)\mu'^*(k,\eta) - \mu^*(k,\eta)\mu'(k,\eta) = i$, so that the canonical commutation relations for the field $h_{ij}(\overline{x})$ and its conjugate momentum imply the usual commutation relations previously stated for the operators $A(\vec{k},l)$ and $A^+(\vec{k},l)$.

In order to solve equation (10), we need first to solve the equations of motion determined by the system of equations (3), (4) and (10) given the equation of state $P = P(\rho)$, from which we obtain the scale factor a, which is the input for the mode equation. At any *transition epoch*—whenever the equation of state of the system varies—the scale factor will in principle be different from the value it had before the transition started; thus the mode equation has to be stated for each transition epoch.

We can formally write down the former as follows: we denote by $\Delta\eta_r$ the transition epoch; we will assume that prior to a time $\eta_r - \Delta\eta_r$, the pressure and density are related by $P_{(r-1)} = F_{(r-1)}(\rho)$ for $\eta < \eta_r - \Delta\eta_r$; during the transition, the form of the function $F_{(r-1)}(\rho)$ changes until it reaches another stable form, $F_{(r)}(\rho)$, so that $P_r = F_{(r)}(\rho)$ for $\eta > \eta_r + \Delta\eta_r$. One can relate the functions Y before and after the transition by means of the so-called *Bogoliubov coefficients,* $\alpha_{(r)}(k)$ and $\beta_{(r)}(k)$, and, from the condition on the Wronskian of the solutions $\mu(k,\eta)$, it follows that they obey $|\alpha_{(r)}(k)|^2 - |\beta_{(r)}(k)|^2 = 1$, so that, for $\eta < \eta_r - \Delta\eta_r$ and $\eta > \eta_r - \Delta\eta_r$, one has $Y_{(r-1)}(k,\eta) = \alpha_{(r)}(k)Y_{(r)}(k,\eta) + \beta_{(r)}(k)Y^*_{(r)}(k,\eta)$.

The creation and annihilation operators prior to and after the transition for each polarization mode can also be related through these coefficients: inserting this last relation into the definition for the $h(\overline{x}, \overrightarrow{k})$ given in equation (9), and using equations (8), (9) for the stages $(r-1)$ and (r), one finds the following relation *before* and *after* the transition respectively (we omit the label l for the two polarization states): $A_{(r-1)}(\overrightarrow{k}) = \alpha^*_{(r)}(k)A_{(r)}(\overrightarrow{k}) - \beta^*_{(r)}(k)A^+_{(r)}(-\overrightarrow{k})$ and $A_{(r)}(\overrightarrow{k}) = \alpha_{(r)}(k)A_{(r-1)}(\overrightarrow{k}) + \beta^*_{(r)}(k)A^+_{(r-1)}(-\overrightarrow{k})$. Thus, the vacuum state of the region $(r-1)$, denoted by $|0_{(r-1)}\rangle$, will be annihilated by $A_{(r-1)}(\overrightarrow{k})$: $A_{(r-1)}(\overrightarrow{k})|0_{(r-1)}\rangle = 0\,\forall\overrightarrow{k}$ but not by $A_{(r)}(\overrightarrow{k})$: $A_{(r)}(\overrightarrow{k})|0_{(r-1)}\rangle = \beta^*_{(r)}(k)A^+_{(r-1)}(-\overrightarrow{k})|0_{(r-1)}\rangle\forall\overrightarrow{k}$, and vice versa. And this is the key to the whole issue: we have spontaneous creation of particles arising from the fact that the vacua before and after the transitions are different.

In order to obtain the above results, two approximations have been considered: the *adiabatic* (as opposed to the *Hamiltonian diagonalization approximation*) and the *sudden transition approximation* (see [2] for a complete discussion on both).

From the above, one defines a *number operator for the mode* $\overrightarrow{k}$ at stage r, $N_{(r)}(\overrightarrow{k}) \equiv A^+_{(r)}(\overrightarrow{k})A_{(r)}(\overrightarrow{k})$ so that we have $\langle 0_{(r-1)}|N_{(r)}(\overrightarrow{k})|0_{(r-1)}\rangle = |\beta_{(r)}(k)|^2$. This expression represents *spontaneous creation* of gravitons from an initial vacuum, but can be generalized to the case of an initial state not being a vacuum (*stimulated creation*).

One can also relate the Bogoliubov coefficients, and thus, the annihilation and creation operators A and A^+, of stages corresponding to different transition epochs; this is useful whenever we are dealing with more than one transition. The relation comes in a recursive expression: $\alpha_{Tr}(k) = \alpha_{(r)}(k)\alpha_{Tr-1}(k) + \beta^*_{(r)}(k)\beta_{Tr-1}(k)$ and $\beta_{Tr}(k) = \beta_{(r)}(k)\alpha_{Tr-1}(k) + \alpha^*_{(r)}\beta_{Tr-1}(k)$ ([3] and [4]).

Once one has solved the mode equation (10), it is possible to find a final general expression for the Bogoliubov coefficients imposing the continuity of the solution and of its first derivative, at each transition epoch. This final expression is given in terms of known and unknown functions $\mu_{(tr)}$ and $R_{(tr)}$ during the transition epoch, but, with the help of the *sudden transition approximation* one can recast these expressions in terms only of known functions; thus

$$\alpha_{(r)}(k) = \mathrm{i}\left[\frac{R_{(r)}(\eta_r)}{R_{(r-1)}(\eta_r)}\alpha^{(0)}_{(r)}(k) + \frac{\mu_{(r-1)}(k,\eta_r)\mu^*_{(r)}(k,\eta_r)}{R_{(r-1)}(\eta_r)}\delta_r\right] \tag{11}$$

$$\beta_{(r)}(k) = \mathrm{i}\left[\frac{R_{(r)}(\eta_r)}{R_{(r-1)}(\eta_r)}\beta^{(0)}_{(r)}(k) - \frac{\mu_{(r-1)}(k,\eta_r)\mu_{(r)}(k,\eta_r)}{R_{(r-1)}(\eta_r)}\delta_r\right] \tag{12}$$

where

$$\alpha^{(0)}_{(r)}(k) = \mu^*_{(r)}(k,\eta_r){\mu'}_{(r-1)}(k,\eta_r) - \mu_{(r-1)}(k,\eta_r)\mu^{*'}_{(r)}(k,\eta_r)$$
$$\beta^{(0)}_{(r)}(k) = \mu_{(r-1)}(k,\eta_r){\mu'}_{(r)}(k,\eta_r) - \mu_{(r)}(k,\eta_r){\mu'}_{(r-1)}(k,\eta_r)$$
$$\delta_r = {R'}_{(r)}(\eta_r) - \frac{R_{(r)}(\eta_r)}{R_{(r-1)}(\eta_r)}{R'}_{(r-1)}(\eta_r).$$

These expressions are completely general with respect to the underlying theory; they are also general in the sense that there is neither any restriction on the actual form of $\mu(k,\eta)$ and $R(\eta)$, apart from the *sudden approximation*. This latter means that the equations will fail whenever $T \lesssim \Delta t_r$ since in that case these short period waves will be sensitive to the details of the transition. There is also an alternative approach to the one outlined here, given in [5, 6].

Once we have obtained the Bogoliubov coefficient β, we can compute the number of gravitons by means of the *number operator* N_k: one obtains the so-called *differential* *energy density* associated with gravitons whose frequency lies in the range between w and $w + \mathrm{d}w$, at any time η and for waves with $\lambda \leqslant \lambda_H$, with λ_H being the *Hubble length,* as $\mathrm{d}\rho_G = P_g(w)\,\mathrm{d}w = 2\hbar w \frac{w^2}{2\pi^2 c^3}\,\mathrm{d}w \langle N_{w(k)}\rangle$, where we have momentarily restored units; the factor of 2 accounts for the two independent polarization states and $\langle N_{w(k)}\rangle = |\beta_{(r)}(k)|^2$ in the spontaneous case. The units of $P_g(w)$ are erg cm^{-3} Hz^{-1}. Integrating the above, one obtains the *total energy density* $\rho_G(\eta) = \int_{w_{\min}(\eta)}^{w_{\mathrm{Max}}(\eta)} P_g(w)\,\mathrm{d}w$ where $w_{\min}(\eta)$ and $w_{\mathrm{Max}}(\eta)$ are appropriate cutoff limits (see [3, 4, 7] for discussions on the topic).

It is worthwhile to point out that in the FLRW flat metric, in the cases where the behaviour of R defined in equation (9) is of the form $R \sim \eta^{\alpha}$—in fact, most cases of interest—the solutions to the mode equation (10) are given in terms of the Hankel functions $H^{(1)}$ and $H^{(2)}$, so that the above general expressions can be greatly simplified [8]. In another arena, a remarkable and most interesting feature was pointed out by L P Grishchuck [9]: the analogy between the mechanism of primordial gravitons production and the phenomena of squeezing in quantum optics, as he noticed that gravitons of relic origin will come in squeezed states, as much as photons do in quantum optics.

3. Application to cosmology

Many observations mentioned before support the existence of the current state of accelerated expansion. Also, different models arise with the aim of explaining the phenomena, among others, the cosmological constant itself, quintessence, and modified or extended gravity theories already mentioned. One recent proposal in this latter line of research are the so-called *phantom scalar–tensor gravity theories* [10].

Most $f(\Re)$ theories aim at explaining the current speed up as a purely gravitational effect, motivated by the fact that positive powers of the curvature added to the standard Einstein–Hilbert Lagrangian may give rise to early-time inflation; thus, in the same way, one aims at explaining the current period of accelerated expansion by the inclusion of terms growing at small values of the curvature (present); this can be accomplished with terms with negative powers of the Ricci scalar. What is remarkable is that these terms may be expected from some time-dependent compactifications of string/M-theory [12].

Thus, there is currently a huge effort in the search of the appropriate form of these terms and there has recently been published a series of important papers [13] establishing the correct form for them, with the important result that, in order for the theory to be compatible with solar system tests, the gravity Lagrangian should be nearly linear in $\Re$ with the possible nonlinearities bounded by quadratic terms at most:

$$\Re - 2\Lambda - \frac{l^2\Re^2}{2} \leqslant f(\Re) \leqslant \Re - 2\Lambda + \frac{l^2\Re^2}{2} \tag{13}$$

(l^2 determines the scale over which nonlinear corrections are relevant).

Our group is currently working in the derivation of graviton spectra from such theories. These graviton spectra can help to shed some light on modified gravity theories once the current generation of gravitational wave detectors will be operating.

There is another interesting application of cosmological graviton production: the reconstruction of the scalar potential of scalar–tensor gravity theories, directly from observations of the graviton emission.

These can be realized by means of the so-called *massive binary black hole inspirals* (MBBH inspirals), which are potentially powerful standard candles for gravitational waves, playing the same role as type Ia supernovae are for electromagnetic waves: observations of their intensity will give the luminosity *distance* D_L and the complementary observation of the electromagnetic counterpart can lead to the distance–redshift relation, which gives the expansion history of the universe and, thus, clues as to the origin of the so far elusive *dark energy*.

These MBBHs sources arise from mergers of galaxies and pregalactic structures at high redshift and have been already observed (ex. NGC 6240). LISA, the laser *interferometer space antenna*, a joint ESA–NASA mission scheduled to be launched within the timeframe of 2015, is expected to measure at least several events of this type [14]. Through this $D_L(z)$ relation then, as shown by Saini, Raychaudhury and Starobinsky [15], one can reconstruct the form of the scalar potential $V(\sigma)$ of a general scalar–tensor gravity theory, and also its equation of state $P/\rho = w$, thus in a model-independent way. This constitutes a way for contrasting the results of the theoretical spectra with observations allowing us to discard and/or fine-tune the parameters of the theory. There also exists a complementary model-independent method of diagnosis of dark energy, called *statefinder* [16].

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